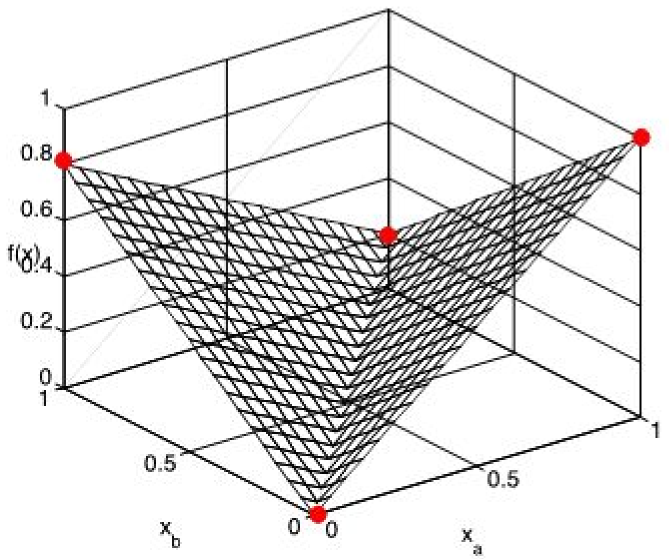
The Lovasz extension

$f^- : [0,1]^V \rightarrow \mathbb{R}_{\geq 0}$ of a submodular function f is given by:

$$f^-(\mathbf{x}) = \mathbb{E}_{\theta \in \mathcal{U}(0,1)}[f(\{i : x_i \geq \theta\})]$$

$$f^-(\mathbf{x}) = \int_{\theta \in \mathcal{U}(0,1)} \theta f(\{i : x_i \geq \theta\})$$



- Continuous extension of submodular function

- Properties:

1. $f^-(\mathbf{1}_S) = f(S)$ for all $S \subseteq V$
   - $\mathbf{1}_S(x) = 1 \text{ if } x \in A; 0 \text{ if } x \notin A$
2. f^- is convex $\Rightarrow \mathbb{E}[f(A)] \geq f(\mathbb{E}[A])$ (Jensen's inequality)
3. $f^-(c \cdot \mathbf{x}) \geq c \cdot f^-(\mathbf{x})$ for any $c \in [0,1]$

These properties give the following lemma:

Lemma 1.

When:

- Let S be a random set
- suppose that $\mathbb{E}[\mathbf{1}_S] = c \cdot \mathbf{p}$ (for $c \in [0,1]$).

Then: $\mathbb{E}[f(S)] \geq c \cdot f^-(p)$.

Proof.

$$\mathbb{E}[f(S)] =_1 \mathbb{E}[f^-(\mathbf{1}_S)] \geq_2 f^-(\mathbb{E}[\mathbf{1}_S]) = f^-(c \cdot \mathbf{p}) \geq_3 c \cdot f^-(\mathbf{p}).$$

Property of Greedy Algorithm

Lemma 2.

When:

- Let $A \subseteq V$ and $B \subseteq V$ be two disjoint subsets of V
- For each element $e \in B$, $\texttt{GREEDY}(A \cup \{e\}) = \texttt{GREEDY}(A)$

Then: $\texttt{GREEDY}(A \cup B) = \texttt{GREEDY}(A)$.

Proof.

Suppose for contradiction that $\texttt{GREEDY}(A \cup B) \neq \texttt{GREEDY}(A)$.
This means $\texttt{GREEDY}(A \cup B)$ contains an element of B.
Let e be the first element of B which is selected by $\texttt{GREEDY}(A \cup B)$.
Then $\texttt{GREEDY}(A \cup B)$ will start from the input $A \cup \{e\}$, which contradicts the fact that $\texttt{GREEDY}(A \cup \{e\}) = \texttt{GREEDY}(A)$.

Preliminaries

Hereditary Constraints

if some set is in $\mathcal{I}$, all of its subsets are in $\mathcal{I}$.

α-approximation

- submodular function subjects to hereditary constraint $\mathcal{I} \subseteq 2^V$
- with any subset $V' \subseteq V$ the algorithm produces a solution $S \subseteq V'$ with $S \in \mathcal{I}$
- $f(S) \geq \alpha \cdot f(\mathrm{OPT})$, where $\mathrm{OPT} \in \mathcal{I}$ is any feasible subset of V'

Why RandGreeDi is $\frac{\alpha}{2}$-approximation?

Fix $\langle V, \mathcal{I}, f \rangle$

- V is input of RandGreeDi
- $\mathcal{I} \subseteq 2^V$ is hereditary constraint
- $f^- : [0,1]^V \to \mathbb{R}_{\geq 0}$ is non-negative, monotone submodular function

Suppose

- Greedy is α-approximation, ALG is β-approximation
- $\mathrm{OPT} = \ \mathrm{argmax}_{A \in \mathcal{I}} f(A)$: a feasible set maximizing f

Let

- $\mathcal{V}(1/m)$ denote the distribution over random subsets of V, where each element is included independently with probability $1/m$.
- $\mathbf{p} \in [0,1]^n$ be the following vector.

For each element $e \in V$, we have

$$p_e = \begin{cases} \Pr_{A \sim \mathcal{V}(1/m)}[e \in \texttt{GREEDY}(A \cup \{e\}) & \text{if } e \in \mathrm{OPT} \\ 0 & otherwise \end{cases}$$

Lemma 3.
For each machine i, $\mathbb{E}[f(S_i)] \geq \alpha \cdot f^{-}(\mathbf{1}_{\mathrm{OPT}} - \mathbf{p})$.

Proof.

Let V_i denote the set of elements assigned to machine i.
Let $O_i = \{e \in \mathrm{OPT} : e \notin \texttt{GREEDY}(V_i \cup \{e\})\}$.

Apply Lemma 2 with $A = V_i$ and $B = Oi \setminus Vi$, then

$$\texttt{GREEDY}(V_i) = \texttt{GREEDY}(V_i \cup O_i) = S_i$$

Since $\mathrm{OPT} \in \mathcal{I}$ and $\mathcal{I}$ is hereditary, $O_i \in \mathcal{I}$. $\texttt{GREEDY}$ is α-approximation, then

$$f(S_i) \geq \alpha \cdot f(O_i)$$

Since the distribution of V_i is the same as $\mathcal{V}(1/m)$, for each element $e \in \mathrm{OPT}$:

$$\mathbf{Pr}[e \in O_i] = 1 - \mathbf{Pr}[e \notin O_i] = 1 - p_e$$

$$\mathbb{E}[\mathbf{1}_{O_i}] = \mathbf{1}_{\mathrm{OPT}} - \mathbf{p}$$

With Lemma 1, we obtain

$$\mathbb{E}[f(S_i)] \geq \alpha \cdot \mathbb{E}[f(O_i)] \geq \alpha \cdot f^{-}(\mathbf{1}_{\mathrm{OPT}} - \mathbf{p})$$

Lemma 4.
$\mathbb{E}[f(\mathrm{ALG}(S))] \geq \beta \cdot f^{-}(\mathbf{p})$.

Proof.

- $S = \bigcup_i \texttt{GREEDY}(V_i)$
- $\mathrm{OPT} \in \mathcal{I}$ and $\mathcal{I}$ is hereditary, $S \cap \mathrm{OPT} \in \mathcal{I}$.
- ALG is β-approximation

We have:

$$f(\mathrm{ALG}(S)) \geq \beta \cdot f(S \cap \mathrm{OPT})$$

Consider an element $e \in \mathrm{OPT}$. For each machine i, we have

$$
\begin{aligned}
&\Pr[e \in S | e \text{ is assigned to machine } i] \\
&= \Pr[e \in \texttt{GREEDY}(V_i) | e \in V_i] \\
&= \Pr_{A \sim \mathcal{V}(1/m)}[e \in \texttt{GREEDY}(A) | e \in A] \\
&= \Pr_{B \sim \mathcal{V}(1/m)}[e \in \texttt{GREEDY}(B \cup \{e\})] \\
&= p_e
\end{aligned}
$$

Therefore,

$$\Pr[e \in S \cap \mathrm{OPT}] = p_e$$

$$\mathbb{E}[\mathbf{1}_{S \cap \mathrm{OPT}}] = \mathbf{p}$$

With Lemma 1,

$$\mathbb{E}[f(\mathrm{ALG}(S))] \geq \beta \cdot \mathbb{E}[f(S \cap \mathrm{OPT})] \geq \beta \cdot f^{-}(\mathbf{p})$$

Theorem. RandGreeDi is an $\frac{\alpha\beta}{\alpha+\beta}$-approximation alogrithm.

Proof.

- $S_i = \texttt{GREEDY}(V_i)$
- $S = \bigcup_i S_i$
- $T = \mathrm{ALG}(S)$

The output D produced by RandGreeDi satisfies:

- $f(D) \geq \max_i(f(S_i))$
- $f(D) \geq f(T)$

From Lemma 3, 4:

- $\mathbb{E}[f(D)] \geq \alpha \cdot f^-(\mathbf{1}_{\text{OPT}} - \mathbf{p})$
- $\mathbb{E}[f(D)] \geq \beta \cdot f^-(\mathbf{p})$

Then, with the fact that f^- is convex and $f^-(c \cdot \mathbf{x}) \geq cf^-(\mathbf{x})$ for any $c \in [0, 1]$:

$$
\begin{aligned}
(\beta + \alpha)\mathbb{E}[f(D)] &\geq \alpha\beta(f^-(p) + f^-(\mathbf{1}_{\text{OPT}} - \mathbf{p})) \\
&\geq \alpha\beta \cdot f^-(\mathbf{1}_{\text{OPT}}) \\
&= \alpha\beta \cdot f(\text{OPT})
\end{aligned}
$$